

COMPACT SUPPORT PROPERTY OF SUPERBROWNIAN MOTION IN RANDOM ENVIRONMENTS

GUOHUAN ZHAO

ABSTRACT. In this paper, we prove the compact support property for a class of nonlinear SPDE including the equation that the density of one-dimensional Super-Brownian Motion in random environment satisfies.

1. INTRODUCTION

It's well know that the one-dimensional density of classic Super-Brownian Motion satisfies the following SPDE,

$$\partial_t u(t, x) = \Delta u(t, x) + \sqrt{u(t, x)} \dot{B}(t, x) \quad (1)$$

Here we replace $\frac{1}{2}\Delta$ with Δ for simple and $\dot{B}(t, x)$ is time-space white noise. The above equation can be written as following form:

$$\partial_t u(t, x) = \Delta u(t, x) + \sqrt{u(t, x)} \varphi_k(x) \dot{\beta}_t^k$$

where $\{\varphi_k\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and $\{\dot{\beta}_t^k\}$ is a sequence of independent Brownian Motions. The solutions to this equation has **compact support property**, roughly speaking, if the initial data $u(0, x)$ have compact support, then for all $t > 0$, $u(t, \cdot)$ has compact support almost surely. In [3], the authors proved the compact support property for the solutions of a large class of SPDEs including (1). Later Krylov given a simpler proof in [2] by using his L^p theory.

On the other hand, from 1990's many experts started to study superprocesses in random environments. In [4], Mytnik introduced models of superprocesses in random environments. We give a brief description below:

Let $\{\xi_k(x), k \in \mathbb{N}\}$ be a sequence of independent identically distributed random fields on \mathbb{R}^d satisfying:

$$\mathbf{E}\xi_k(x) = 0, \quad \mathbf{E}\xi_k(x)\xi_k(y) = g(x, y), \quad \sup_x \mathbf{E}|\xi_k(x)|^3 < \infty, \quad x \in \mathbb{R}^d, k \in \mathbb{N}. \quad (2)$$

Where g is the covariance function which satisfies $\sup_{x, y \in \mathbb{R}^d} |g(x, y)| \leq C < \infty$, $g(x, \cdot) \in C_0(\mathbb{R}^d)$ ($\forall x \in \mathbb{R}^d$), we further assume $g \in C_b^2$ in this paper. $\{\xi_k(x), k \in \mathbb{N}\}$ serves as the random environments. For each fixed $n \in \mathbb{N}$, consider a particle system in which there are $K_n \geq 1$ particles located in \mathbb{R}^d , each of them moves independent as a copy of Brownian motion (with generator Δ) until time $t = 1/n$. Given $\{\xi_k(x), k \in \mathbb{N}\}$, at time $\frac{1}{n}$, each particle split into two particles with probability $\frac{1}{2} + \frac{1}{2\sqrt{n}}[(-\sqrt{n}) \vee \xi_1(x) \wedge (\sqrt{n})]$ or dies with probability $\frac{1}{2} - \frac{1}{2\sqrt{n}}[(-\sqrt{n}) \vee \xi_1(x) \wedge (\sqrt{n})]$. The new particles then moves in space independently as Brownian motions (with generator Δ) starting at their place of birth, during the time interval $[1/n, 2/n]$. In general, at time $\frac{i}{n}$, each surviving particle split into two particles with probability $\frac{1}{2} + \frac{1}{2\sqrt{n}}[(-\sqrt{n}) \vee \xi_i(x) \wedge (\sqrt{n})]$ or dies with

probability $\frac{1}{2} - \frac{1}{2\sqrt{n}}[(-\sqrt{n}) \vee \xi_i(x) \wedge (\sqrt{n})]$, and in the time interval $[i/n, (i+1)/n]$ particles independently according to Brownian motions (with generator Δ). Let X_t^n be the measure-valued Markov process, defined as

$$X_t^n(B) = \frac{\text{number of particles in } B \text{ at time } t}{n}.$$

where $B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets in \mathbb{R}^d . Let $C_b^k(\mathbb{R}^d)$ (respectively $C_b^\infty(\mathbb{R}^d)$) denote the collection of all bounded continuous functions on \mathbb{R}^d with bounded continuous derivatives up to order k (respectively with bounded derivatives of all orders). For all bounded measurable f , let $\mu(f) = \langle f, \mu \rangle$ denote the integral of f with respect to the measure μ on \mathbb{R}^d . For any measurable functions f, h on \mathbb{R}^d , let $f \otimes h$ denote an \mathbb{R}^{2d} valued function defined by $(f \otimes h)(x, y) := f(x)h(y)$, $x, y \in \mathbb{R}^d$. It was proved in [5] that if X_0^n converge weakly to a finite measure μ on \mathbb{R}^d , then the processes $X^n = \{X_t^n, t \geq 0\}$ converges weakly to a measure-valued process $X = \{X_t, t \geq 0\}$, where X is the unique solution to the following martingale problem:

$$(MP) : \begin{cases} \text{for all } f \in C_b^2, & M_t^f = \langle f, X_t \rangle - \langle f, \mu \rangle - \int_0^t \langle \Delta f, X_s \rangle \\ & \text{is a continuous square-integrable martingale with quadratic variation} \\ \langle M^f \rangle_t = \int_0^t \langle f^2, X_s \rangle ds + \int_0^t \langle g \cdot f \otimes f, X_s \otimes X_s \rangle ds. \end{cases} \quad (3)$$

However, there only very few works about the properties of these processes. As far as to my knowledge, the most interesting work is [5], in which the authors studies the local extinction property. Under some assumption, it shows the super-Brownian motion in random environments will extinct locally in any dimension which is quite different from the classic case (see Theorem 1.1, 1.2 of [5] for more details).

In this paper, we show the density of one-dimensional super-Brownian motion in random environments satisfies:

$$\partial_t u = \Delta u + \sqrt{u} \varphi_k \dot{\beta}_t^k + u h_k \dot{w}_t^k.$$

Here $\{w_t^k\}$ and $\{\beta_t^k\}$ are two sequences of independent standard Brownian motions. In order to do that we give the second order moment formula of superprocesses in Section 2. In Section 3, using Krylov's L^p theory for SPDEs (see [1]), we prove the compact support property for more general stochastic partial differential equations under some reasonable assumptions.

2. MOMENT FORMULA

In this section, we give the first and second order moment formula for X_t .

The following Lemma is a simple application of Stone-Weierstrass theorem.

Lemma 2.1. *For any $f(x, y) \in C_b(\mathbb{R}^{d_1+d_2})$ ($x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}$), there exist a sequence of smooth functions $\{f_n(x, y)\}$ with form $f_n(x, y) = \sum_{i=1}^n \phi_i^n(x) \psi_i^n(y)$ such that $f_n \rightarrow f$ uniformly on any compact subset in $\mathbb{R}^{d_1+d_2}$ and $\|f_n\|_\infty \leq \|f\|_\infty$.*

In order to get the second moment formula, we need the following generalization of Lemma 2.1.

Lemma 2.2. *For any $f(x, y) \in C_b^{k_1, k_2}(\mathbb{R}^{d_1+d_2})$ ($x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}$), there exist a sequence of smooth function $\{f_n(x, y)\}$ with form $f_n(x, y) = \sum_{i=1}^n \phi_i^n(x) \psi_i^n(y)$ such that for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^{d_1}$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^{d_2}$ and $|\alpha| = \alpha_1 + \dots + \alpha_d \leq k_1$, $|\beta| = \beta_1 + \dots + \beta_d \leq k_2$. $\partial_x^\alpha \partial_y^\beta f_n \rightarrow \partial_x^\alpha \partial_y^\beta f$ uniformly on any compact subset and $\|f_n\|_{C^{k_1, k_2}} \leq C\|f\|_{C^{k_1, k_2}}$.*

Proof. To keep the proof simple, we assume $k_i, d_i = 1$. Let $\varphi(x) \in C_c^\infty(\mathbb{R})$ and $\varphi(x) = 1$ if $|x| \leq 1/2$, $\varphi(x) = 0$ if $|x| \geq 1$. Fixed $R > 0$, let $f_R(x, y) = f(x, y)\varphi(\frac{x}{R})\varphi(\frac{y}{R})$. Using standard diagonal argument, in order to prove the Lemma, we only need to show there exists a sequence of smooth functions $f_{R,n}(x, y) = \sum_{i=1}^n \phi_i^n(x) \psi_i^n(y)$ such that $\|f_{R,n} - f_R\|_{C^1} \rightarrow 0$ ($n \rightarrow \infty$). Now define

$$f_R^m(x, y) = \int_{\mathbb{R}^2} m^2 \varphi(m\xi) \varphi(m\eta) f_R(x - \xi, y - \eta) d\xi d\eta \in C_c^\infty(\mathbb{R}^2).$$

Since $f_R \in C_c^1$, we have

$$\lim_{m \rightarrow 0} \|f_R^m - f_R\|_{C^1} = 0. \quad (4)$$

On the other hand, we have

$$f_R^m(x, y) = \int_{-R}^y \int_{-R}^x \partial_x \partial_y f_R^m(\xi, \eta) d\xi d\eta,$$

by Lemma 2.1, there exist a sequence of smooth functions $g_R^{m,n} = \sum_{i=1}^n \phi_i^{m,n}(x) \psi_i^{m,n}(y)$ such that $g_R^{m,n}(x, y) \rightarrow \partial_x \partial_y f_R^m$ uniformly as $n \rightarrow \infty$. Let $f_R^{m,n} = \int_{-R}^y \int_{-R}^x g_R^{m,n}(\xi, \eta) d\xi d\eta$, then

$$\lim_{n \rightarrow \infty} \|f_R^{m,n} - f_R^m\|_{C^1} = 0. \quad (5)$$

Combining (4) and (5), using standard diagonal argument, we can find a sequence of function $f_{R,n} = \sum_{i=1}^n \phi_i^n(x) \psi_i^n(y) \rightarrow f_R$ in C^1 . \square

Before giving the second moment formula, we first prove an estimate for $\mathbf{E}_\mu \langle f, X_t \rangle^2$.

Lemma 2.3.

$$\begin{aligned} \mathbf{E}_\mu \langle f, X_t \rangle &= \langle P_t f, \mu \rangle; \\ \mathbf{E}_\mu \langle f, X_t \rangle^2 &\leq \left\{ \langle P_t f, \mu \rangle^2 + \int_0^t \langle \mu, P_s [(P_{t-s} f)^2] \rangle ds \right\} \exp(\|g\|_\infty t). \end{aligned}$$

Where $\{P_t\}$ is the semigroup whose generator is Δ .

Proof. Just as the proof of Proposition II 5.7 of [6], if $\phi_t(x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, we can prove

$$\langle \phi_t, X_t \rangle - \langle \phi_0, \mu \rangle - \int_0^t \langle \dot{\phi}_s + \Delta \phi_s, X_s \rangle ds$$

is a martingale with quadratic variation $\langle M \rangle_t = \int_0^t \langle \phi_s^2, X_s \rangle ds + \int_0^t \langle g \cdot \phi_s \otimes \phi_s, X_s \otimes X_s \rangle ds$. In addition, there exists a martingale measure $M(dt, dx)$ such that for any $f \in \mathcal{B}(\mathbb{R}^d)$,

$$\langle f, X_t \rangle = \langle \mu, P_t f \rangle + \int_0^t \int_{\mathbb{R}^d} P_{t-s} f(x) M(ds, dx)$$

$\left\{ \int_0^s p_{t-r} f(x) M(dr, dx) \right\}_{s \leq t}$ is a martingale from time 0 to t with

$$\begin{aligned} \left\langle \int_0^s P_{t-r} f(x) M(dr, dx) \right\rangle_s &= \int_0^s \int_{\mathbb{R}^d} (P_{t-r} f(x))^2 X_r(dx) dr \\ &\quad + \int_0^s \int_{\mathbb{R}^{2d}} g(x, y) P_{t-r} f(x) P_{t-r} f(y) X_r(dx) X_r(dy) dr \end{aligned}$$

Hence

$$\mathbf{E}_\mu \langle f, X_t \rangle = \langle P_t f, \mu \rangle,$$

$$\begin{aligned} \mathbf{E}_\mu [\langle P_{t-s} f, X_s \rangle^2] &= [\langle P_t f, \mu \rangle]^2 + \mathbf{E}_\mu \int_0^s \int_{\mathbb{R}^d} (P_{t-r} f(x))^2 X_r(dx) dr \\ &\quad + \mathbf{E}_\mu \int_0^s \int_{\mathbb{R}^{2d}} g(x, y) P_{t-r} f(x) P_{t-r} f(y) X_r(dx) X_r(dy) dr \\ &\leq [\langle P_t f, \mu \rangle]^2 + \int_0^s \langle \mu, P_r [(P_{t-r} f)^2] \rangle dr + \|g\|_\infty \int_0^s \mathbf{E}_\mu [\langle P_{t-r} f, X_r \rangle^2] ds \end{aligned}$$

By Gronwall's Inequality, we obtain

$$E_\mu [\langle f, X_t \rangle^2] \leq \left\{ [\langle P_t f, \mu \rangle]^2 + \int_0^t \langle \mu, P_r [(P_{t-s} f)^2] \rangle ds \right\} \exp(\|g\|_\infty t)$$

□

Now we are in a position to prove the second moment formula:

Theorem 2.4. *Let $Q_t = P_t \otimes P_t$ and Q_t^g be the semigroup generated by $\Delta + g$. Then for all $\phi \in C_b(\mathbb{R}^d)$,*

$$\mathbf{E}_\mu (X_t(\phi)^2) = \langle Q_t^g(\phi \otimes \phi), \mu \otimes \mu \rangle + \int_0^t \langle P_s(\pi Q_{t-s}^g(\phi \otimes \phi)) \rangle ds, \mu \rangle$$

Here

$$(\pi f)(x) = f(x, x); \quad f \in \mathcal{B}(\mathbb{R}^{2d}), \quad x \in \mathbb{R}^d.$$

Proof. We assume $\mu = \delta_0$ for simple. The proof for general $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ is similar.

$\forall \phi_s, \psi_s \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, using Ito's Formula,

$$d(X_s(\phi_s)X_s(\psi_s)) = X_s(\phi_s)d(M_s^\psi + V_s^\psi) + X_s(\psi_s)d(M_s^\phi + V_s^\phi) + d\langle M^\phi, M^\psi \rangle_s$$

Where

$$V_s^\phi = \langle \phi, \mu \rangle + \int_0^s \langle \dot{\phi}_r + \Delta \phi_r, X_r \rangle dr, \quad V_s^\psi = \langle \psi, \mu \rangle + \int_0^s \langle \dot{\psi}_r + \Delta \psi_r, X_r \rangle dr.$$

$$M_s^\phi = \langle \phi_s, X_s \rangle - V_s^\phi, \quad M_s^\psi = \langle \psi_s, X_s \rangle - V_s^\psi;$$

taking expectation,

$$\begin{aligned}
 \mathbf{E}_{\delta_0}(X_t(\phi_t)X_t(\psi_t)) &= \phi_0(0)\psi_0(0) + \mathbf{E}_{\delta_0} \int_0^t X_s(\phi_s)dV_s^\psi + \mathbf{E}_{\delta_0} \int_0^t X_s(\psi_s)dV_s^\phi + \mathbf{E}_{\delta_0}\langle M^\phi, M^\psi \rangle_t \\
 &= \phi_0(0)\psi_0(0) + \mathbf{E}_{\delta_0} \int_0^t [X_s(\phi_s)X_s(\dot{\psi}_s + \Delta\psi_s) + X_s(\psi_s)X_s(\dot{\phi}_s + \Delta\phi_s)]ds \\
 &\quad + X_s \otimes X_s(g \cdot \phi_s \otimes \psi_s)ds + \mathbf{E}_{\delta_0} \int_0^t X_s(\phi_s\psi_s)ds \\
 &= \phi_0(0)\psi_0(0) + \mathbf{E}_{\delta_0} \int_0^t X_s \otimes X_s(\partial_s(\phi_s \otimes \psi_s) + (\Delta + g)(\phi_s \otimes \psi_s))ds \\
 &\quad + \mathbf{E}_{\delta_0} \int_0^t X_s(\pi\phi_s \otimes \psi_s)ds
 \end{aligned}$$

By linearity, identity

$$\mathbf{E}_{\delta_0}(X_t \otimes X_t(f(t))) = f(0, 0, 0) + \mathbf{E}_{\delta_0} \int_0^t X_s \otimes X_s(\dot{f}(s) + \Delta f(s) + gf(s))ds + \mathbf{E}_{\delta_0} \int_0^t X_s(\pi f(s))ds \quad (6)$$

holds for all $f(t, x, y) = \sum_{i=1}^n \phi_i(t, x)\psi_i(t, y)$.

$\forall f(s, x, y) \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{2d})$. By Lemma 2.2, we can find functions $f^n(s, x, y) = \sum_{i=1}^n \phi_i^n(s, x)\psi_i^n(s, y)$ such that for any fixed $R > 0$,

$$\lim_{n \rightarrow \infty} \|f^n - f\|_{C^{1,2}(Q_R)} = 0$$

and $\|f^n\|_{C_b^{1,2}} \leq C\|f\|_{C_b^{1,2}}$. Here $Q_R = [0, R] \times [-R, R]^{2d}$ and C is independent with n .

Let $\delta^n(s, x, y) = f(s, x, y) - f^n(s, x, y)$, then $\lim_{n \rightarrow \infty} \|\delta^n\|_{C^{1,2}(Q_R)} = 0$ and $\|\delta^n\|_{C_b^{1,2}} \leq C\|f\|_{C_b^{1,2}}$, here C is independent with n . Define $I_R(x) = I_{[-R, R]^d}(x)$, $I'_R(x) = 1 - I_R(x)$. we get

$$|\delta^n(s, x, y)| \leq C(I'_R(x)I'_R(y) + I'_R(x)I_R(y) + I_R(x)I'_R(y)) + \|\delta^n\|_{C^{1,2}(Q_R)}I_R(x)I_R(y).$$

Using Lemma 2.3,

$$\begin{aligned}
 \mathbf{E}_{\delta_0}X_s \otimes X_s(|\delta^n(s)|) &\leq C\mathbf{E}_{\delta_0}(X_s(I_R)X_s(I'_R) + X_s(I'_R)^2) + c_n\mathbf{E}_{\delta_0}X_s(I_R)^2 \\
 &\leq C\left\{\mathbf{E}_{\delta_0}X_s(I'_R)^2 + [\mathbf{E}_{\delta_0}X_s(I'_R)^2]^{\frac{1}{2}}[\mathbf{E}_{\delta_0}X_s(I_R)^2]^{\frac{1}{2}}\right\} \\
 &\quad + c_n\mathbf{E}_{\delta_0}X_s(I_R)^2.
 \end{aligned} \quad (7)$$

Here $c_n = \|\delta^n\|_{L^\infty(Q_R)} \rightarrow 0$ ($n \rightarrow \infty$).

$\mathbf{E}_{\delta_0}X_s(I_R)^2 \leq \mathbf{E}_{\delta_0}X_s(1)^2 \leq C$ and again by Lemma 2.3,

$$\mathbf{E}_{\delta_0}X_s(I'_R)^2 \leq C\left\{[P_s I'_R]^2(0) + \int_0^s P_r[(P_{s-r}I'_R)^2](0)dr\right\} \quad (8)$$

For any $\epsilon > 0$, choose R so large, such that $\frac{1}{(\sqrt{2\pi t})^d} \int_{|y| > \frac{R}{2}} e^{-|y|^2/2t} dy < \epsilon^2$. Then

$$\sup_{s \leq t} [P_s I'_R]^2(0) \leq \left(\frac{1}{\sqrt{2\pi t}} \int_{|x| > R} e^{-x^2/2t} dx\right)^2 < \epsilon^2. \quad (9)$$

$$[(P_{s-r}I'_R)^2](x) \leq \left(I_{\frac{R}{2}}(x) \int_{|y|>R} e^{|x-y|^2/2(s-r)} dy + I'_{\frac{R}{2}}(x) \right)^2 \leq C\epsilon^2 I_{\frac{R}{2}}(x) + CI'_{\frac{R}{2}}(x)$$

Hence

$$\int_0^s P_r[(P_{s-r}I'_R)^2](0)dr \leq C\epsilon^2 s + C \int_0^s \frac{1}{\sqrt{2\pi r}} \int_{|y|>\frac{R}{2}} e^{|y|^2/2r} dy dr \leq C\epsilon^2 \quad (10)$$

So by (7), (8), (9), (10), we have

$$\lim_{n \rightarrow \infty} \sup_{s \leq t} \mathbf{E}_{\delta_0} X_s \otimes X_s (|\delta^n(s)|) = 0. \quad (11)$$

By the same argument we can prove

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \mathbf{E}_{\delta_0} \int_0^t X_s \otimes X_s (\partial_s f^n(s) + (\Delta + g)f^n(s)) ds - \mathbf{E}_{\delta_0} \int_0^t X_s \otimes X_s (\partial_s f(s) + (\Delta + g)f(s)) ds \right| \\ & \leq \lim_{n \rightarrow \infty} \int_0^t \mathbf{E}_{\delta_0} X_s \otimes X_s (|\partial_s \delta^n(s) + (\Delta + g)\delta^n(s)|) ds \\ & \leq C \|f\|_{C^{1,2}([0,t] \times \mathbb{R}^2)} \lim_{R \rightarrow \infty} \int_0^t \mathbf{E}_{\delta_0} (X_s(I_R)X_s(I'_R) + X_s(I'_R)^2) ds \\ & \quad + \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \|\delta^n\|_{C^{1,2}(Q_R)} \int_0^t \mathbf{E}_{\delta_0} X_s(I_R)^2 ds \\ & = 0 \end{aligned} \quad (12)$$

Similarly,

$$\lim_{n \rightarrow \infty} \left| \mathbf{E}_{\delta_0} \int_0^t X_s (\pi f^n(s)) ds - \mathbf{E}_{\delta_0} \int_0^t X_s (\pi f(s)) ds \right| = 0. \quad (13)$$

So we obtain (6) holds for all $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{2d})$.

Suppose $\phi \in C_c^\infty$, let $f_s = Q_{t-s}^g \phi \otimes \phi$ (define $f_s = f_t$ if $s > t$), then $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^2)$. Hence for all $\phi \in C_c^\infty(\mathbb{R})$, we have the formula

$$\mathbf{E}_{\delta_0}(X_t(\phi)^2) = (Q_t^g(\phi \otimes \phi))(0, 0) + \int_0^t [P_s(\pi Q_{t-s}^g(\phi \otimes \phi))](0) ds.$$

A simple approximation argument shows for all $\phi \in C_b(\mathbb{R}^d)$

$$\mathbf{E}_{\delta_0}(X_t(\phi)^2) = \langle Q_t^g(\phi \otimes \phi), \delta_0 \otimes \delta_0 \rangle + \int_0^t \langle P_s(\pi Q_{t-s}^g(\phi \otimes \phi)) ds, \delta_0 \rangle$$

□

Remark 2.5. Indeed, we can also using conditional Laplace transform introduced by [5] to get the same formula. However, the proof presented here is more elementary.

3. COMPACT PROPERTY

In this section, we first use the moment formula to get the equation that the density of Super-Brownian motion satisfies and then prove the compact support property for a class of parabolic SPDEs.

Lemma 3.1. *Suppose $\mu(\mathbb{R}) < \infty$, then the density of the 1-d Super-Brownian Motion in random environments satisfies the following SPDE:*

$$\partial_t u = \Delta u + \sqrt{u} \varphi_k \dot{\beta}_t^k + u h_k \dot{w}_t^k$$

in weak sense, which means for any $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) u(t, x) dx &= \int_{\mathbb{R}} \phi(x) u(0, x) + \int_0^t \int_{\mathbb{R}} u(s, x) \phi''(x) dx ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}} \phi(x) \varphi_k(x) \sqrt{u(s, x)} dx d\beta_s^k + \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}} \phi(x) h_k(x) u(s, x) dx dw_s^k. \end{aligned} \quad (14)$$

Where β^k, w^k are independent Brownian Motion.

Proof. By the second moment formula, we have

$$\mathbf{E}_\mu(\langle X_t, \phi \rangle \langle X_t, \psi \rangle) = (\mu \otimes \mu)(Q_t^g(\phi \otimes \psi)) + \int_0^t \mu[P_s(\pi Q_{t-s}^g(\phi \otimes \psi))]$$

Denote q_t^g be the density of Q_t^g . Using the above equation,

$$\begin{aligned} &\mathbf{E}_\mu(\langle X_t, p(\epsilon, x - \cdot) \rangle \langle X_t, p(\epsilon', x - \cdot) \rangle) \\ &= \int_{\mathbb{R}^2} \mu(y_1) \mu(y_2) \int_{\mathbb{R}^2} q_t^g((y_1, y_2), (z_1, z_2)) p(\epsilon, x - z_1) p(\epsilon', x - z_2) dz_1 dz_2 \\ &\quad + \int_0^t ds \int_{\mathbb{R}} \mu(dw) \int_{\mathbb{R}^3} p(s, w - y) q_{t-s}^g((y, y), (z_1, z_2)) p(\epsilon, x - z_1) p(\epsilon', x - z_2) dz_1 dz_2 dy \\ &= I(t, x) + II(t, x) \end{aligned} \quad (15)$$

It's not hard to prove that

$$\begin{aligned} &\int_0^T dt \int_{\mathbb{R}} I(t, x) dx \\ &= \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}^2} \mu(dy_1) \mu(dy_2) \int_{\mathbb{R}^2} q_t^g((y_1, y_2), (z_1, z_2)) p(\epsilon, x - z_1) p(\epsilon', x - z_2) dz_1 dz_2 \\ &\xrightarrow{\epsilon, \epsilon' \rightarrow 0} \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}^2} \mu(dy_1) \mu(dy_2) \int_{\mathbb{R}^2} q_t^g((y_1, y_2), (x, x)) \\ &\leq C \int_0^T dt \int_{\mathbb{R}^2} \mu(dy_1) \mu(dy_2) \int_{\mathbb{R}} p(t, x - y_1) p(t, x - y_2) dx \\ &\leq C \mu(\mathbb{R})^2 \int_0^T \frac{1}{\sqrt{t}} dt < \infty \end{aligned} \quad (16)$$

By the same argument,

$$\begin{aligned} \int_0^T dt \int_{\mathbb{R}} II(t, x) dx &\xrightarrow{\epsilon, \epsilon' \rightarrow 0} \int_0^T dt \int_0^t ds \int_{\mathbb{R}} \mu(dw) \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} p(s, w - y) q_{t-s}^g((y, y), (x, x)) dy \\ &\leq C \mu(\mathbb{R}) \int_0^T dt \int_0^t \frac{1}{\sqrt{t-s}} ds < \infty \end{aligned} \quad (17)$$

By (15), (16), (17), we get if $\mu(\mathbb{R}) < \infty$, $\{\langle p(\epsilon, x - \cdot), X_t \rangle\}$ is a Cauchy sequence in $L^2(\Omega \times [0, T] \times \mathbb{R})$, define

$$u_t(x) = \lim_{\epsilon \rightarrow 0} \langle p(\epsilon, x - \cdot), X_t \rangle$$

we have

$$M_t^\phi = (u_t, \phi) - (u_0, \phi) - \int_0^t (u_s, \Delta \phi) ds$$

is a martingale with quadratic variation

$$\langle M^\phi \rangle_t = \int_0^t (\phi^2, u_s) ds + \int_0^t ds \int_{\mathbb{R}^2} \sum_k h_k(x) h_k(y) \phi(x) \phi(y) dx dy$$

By martingale representation theorem, there exist independent Brownian sheet $B(t, x)$ and time-white, space-colored Gaussian noise $W(t, x)$ with $\mathbf{E}(W(t, x)W(s, y)) = (s \wedge t)g(x, y)$ such that

$$M_t^\phi = \int_0^t \int_{\mathbb{R}} \phi(x) \sqrt{u_s(x)} B(ds, dx) + \int_0^t \int_{\mathbb{R}} \phi(x) u_s(x) W(ds, x) dx$$

The rightside of above equation can be written as

$$\int_0^t (\phi, \sqrt{u} \varphi_k) d\beta_s^k + \int_0^t (\phi, u_s h_k) dw_s^k$$

where $\{\varphi_k\}$ is a orthonormal basis of $L^2(\mathbb{R})$, $\sum_k h_k(x) h_k(y) = g(x, y)$ and β_t^k, w_t^k are independent Brownian motions. Hence

$$(u_t, \phi) = (u_0, \phi) + \int_0^t (u_s, \phi'') ds + \int_0^t (\sqrt{u_s} \varphi_k, \phi) d\beta_s^k + \int_0^t (u_s h_k, \phi) dw_s^k$$

□

Now we begin to consider the compact support property of following parabolic SPDEs:

$$\begin{aligned} \partial_t u &= \Delta u + u^\gamma \varphi_k \dot{\beta}_t^k + u h_k \dot{w}_t^k \\ (u_t, \phi) &= (u_0, \phi) + \int_0^t (u_s, \phi'') ds + \int_0^t (u_s^\gamma \varphi_k, \phi) d\beta_s^k + \int_0^t (u_s h_k, \phi) dw_s^k \end{aligned} \quad (18)$$

Here $\gamma \in [1/2, 1)$, $\{\varphi_k\}$ is the standard orthonormal basis of $L^2(\mathbb{R})$, $\{h_k\}$ satisfies

$$\sup_{x \in \mathbb{R}} \sum_k h_k^2(x) < \infty.$$

Define

$$C_{tem} = \left\{ f \in C(\mathbb{R}) : \int_{\mathbb{R}} e^{-\lambda|x|} |f(x)| dx < \infty; \forall \lambda > 0 \right\}$$

The next lemma is standard.

Lemma 3.2. *If $u_t(x) \in C_{tem}$ is the weak solution to Equation (18) with initial data $u_0 \in C_{tem}$, then $u_t(x)$ satisfies the following equation:*

$$\begin{aligned} u_t(x) = & p_t * u_0(x) + \int_0^t \left[\int_{\mathbb{R}} p_{t-s}(x-y) u_s^\gamma(y) \varphi_k(y) dy \right] d\beta_s^k \\ & + \int_0^t \left[\int_{\mathbb{R}} p_{t-s}(x-y) u_s(y) h_k(y) dy \right] dw_s^k \end{aligned} \quad (19)$$

Lemma 3.3. *Suppose $u_0 \in C_{tem}^+$, there exists an $\{\mathfrak{B}_t\}$ -space-time white noise $\dot{B}(t, x)$, an independent $\{\mathfrak{W}_t\}$ -time white space colored noise $W(t, x)$ and a $C(\mathbb{R}_+; C_{tem}^+) \cap C(\mathbb{R}_+ \times \mathbb{R})$ solution $u(t, \cdot) \in \mathfrak{F}_t = \mathfrak{B}_t \vee \mathfrak{W}_t$ to (18) on a suitable probability space with filtration $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbf{P})$. What's more, for any $\lambda > 0$,*

$$\sup_{t \leq T} \mathbf{E} \int_{\mathbb{R}} |u_t(x)|^p e^{-\lambda|x|} dx \leq C(T, \lambda) \left\{ 1 + \int_{\mathbb{R}} u_0^p(x) e^{-\lambda|x|} dx \right\} \quad (20)$$

Proof. The proof for existence of C_{tem}^+ solution to (18) is similar with Theorem 2.6 in [7], so we only prove (20) here.

Taking p 's power in both side of (19) then taking expectation, using BDG inequality and Minkowski inequality, we obtain

$$\begin{aligned} \mathbf{E}|u_t(x)|^p & \leq C \left\{ |p_t * u_0|^p + \mathbf{E} \left[\int_0^t ds \int_{\mathbb{R}} p_{t-s}^2(x-y) u_s^{2\gamma}(y) dy \right]^{p/2} \right. \\ & \quad \left. + \mathbf{E} \left[\int_0^t ds \sum_k \left(\int_{\mathbb{R}} p_{t-s}(x-y) u_s(y) h_k(y) dy \right)^2 \right]^{p/2} \right\} \\ & \leq C \left\{ |p_t * u_0|^p + \mathbf{E} \left[\int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y) (1 + u_s^2(y)) dy \right]^{p/2} \right. \\ & \quad \left. + \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-y) \mathbf{E} u_s^p(y) dy \right\} \\ & \leq C \left\{ |p_t * u_0|^p + \int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y) (1 + (\mathbf{E} u_s^p(y))) dy \right. \\ & \quad \left. + \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-y) \mathbf{E} u_s^p(y) dy \right\} \end{aligned}$$

Hence, for any $t \leq T$ we have

$$\begin{aligned} \int_{\mathbb{R}} e^{-\lambda|x|} \mathbf{E}|u_t(x)|^p dx & \leq C \left\{ 1 + \int_{\mathbb{R}} u_0^p(y) dy \int_{\mathbb{R}} p_t(x-y) e^{-\lambda|x|} dx \right. \\ & \quad + \int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} \mathbf{E} u_s^p(y) dy \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y) e^{-\lambda|x|} dx \\ & \quad \left. + \int_0^t ds \int_{\mathbb{R}} \mathbf{E} u_s^p(y) dy \int_{\mathbb{R}} p_{t-s}(x-y) e^{-\lambda|x|} dx \right\} \\ & \leq C \left\{ 1 + \int_{\mathbb{R}} u_0^p(x) e^{-\lambda|x|} dx + \int_0^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}} e^{-\lambda|y|} u_s^p(y) dy \right\} \end{aligned}$$

In the last inequality, we use the element inequality: $\sup_{t \leq T} e^{\lambda|y|} \int_{\mathbb{R}} p_t(x-y) e^{-\lambda|x|} dx \leq C$. Denote $f(t) = \sup_{s \leq t} \int_{\mathbb{R}} e^{-\lambda|x|} \mathbf{E}|u_s(x)|^p dx$, $A = 1 + \int_{\mathbb{R}} u_0^p(x) e^{-\lambda|x|} dx$ then,

$$\begin{aligned} f(t) &\leq CA + C \int_0^t f(s) \frac{ds}{\sqrt{t-s}} \leq CA + C \int_0^t \frac{ds}{\sqrt{t-s}} \left(CA + C \int_0^s f(r) \frac{dr}{\sqrt{s-r}} \right) \\ &\leq CA + C \int_0^t f(r) dr \int_r^t \frac{ds}{\sqrt{(t-s)(s-r)}} \\ &\leq CA + C \int_0^t f(s) ds \end{aligned}$$

Using Gronwall's inequality, we obtain (20). \square

Corollary 3.4. *Suppose $u \in C(\mathbb{R}_+; C_{tem}^+)$ is a solution to (18) with $u_0(x) = 0$ ($x \geq 0$) then for any $T > 0$,*

$$a_p(T) \triangleq \sup_{n \in \mathbb{N}} \mathbf{E} \int_0^T dt \int_n^{n+1} u_t^p(x) dx < \infty$$

Proof. For any $n \in \mathbb{N}$, let $v_t(x) = u_t(n+x) \in C([0, T], C_{tem}^+)$, v satisfies the equation

$$\partial_t v_t(x) = \Delta v_t(x) + v_t^\gamma(x) \varphi_k(n+x) \dot{\beta}_t^k + v_t(x) h_k(n+x) \dot{w}_t^k$$

Since $\{\varphi_k(n+\cdot)\}$ is again the orthonormal basis of $L^2(\mathbb{R})$, $\{h_k(n+\cdot)\}$ satisfies the some condition with $\{h_k\}$. By Lemma (3.3) we have

$$\begin{aligned} E \int_0^T \int_0^1 v_t^p(x) dx dt &\leq C(T) \left\{ 1 + \int_{-\infty}^0 e^{-|x|} u_0(n+x) dx \right\} \\ &\leq C(T) \left\{ 1 + \int_{-\infty}^{-n} e^{-|x|} u_0(x) dx \right\} \\ &\leq C(T) \end{aligned}$$

The last constant is independent with n . \square

The following theorem is our main result.

Theorem 3.5. *If $u_0 \in C_{tem}^+$, $u_0(x) = 0$ ($x \geq 0$), $u_t(x) \in C(\mathbb{R}_+, C_{tem}^+)$ is the solution to equation (18), then $\exists N(\omega)$, such that $u(t, x, \omega) = 0 \forall x \geq N(\omega)$.*

Before proving the main theorem, we need some simple estimates. Define

$$P_t f(x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-|x-y|^2/4t} f(y) dy,$$

then

$$\mathcal{F} \left(\int_0^\infty t^{\delta/2-1} e^{-t} P_t f dt \right) (\xi) = \mathcal{F} f(\xi) \int_0^\infty t^{\delta/2-1} e^{-t} e^{-t|\xi|^2} dt = \mathcal{F} f(\xi) \Gamma(\delta/2) (1 + |\xi|^2)^{-\delta/2}$$

Hence

$$(1 - \Delta)^{-\delta/2} f = c(\delta) \int_0^\infty t^{\delta/2-1} e^{-t} P_t f dt$$

Define

$$R_\delta(x) = c(\delta) \int_0^\infty t^{\delta/2-1} (2\pi t)^{-1/2} e^{-t} e^{-|x|^2/4t} dt = c(\delta) \int_0^\infty t^{\delta/2-3/2} e^{-t} e^{-|x|^2/4t} dt$$

Suppose $\delta < 1$, if $|x| \ll 1$ then

$$\int_0^\infty t^{\delta/2-3/2} e^{-t} e^{-|x|^2/4t} dt \leq C|x|^{\delta-1} \int_0^\infty s^{-\delta/2-1/2} e^{-s} ds \leq C|x|^{\delta-1}$$

If $|x| \gg 1$,

$$\int_0^\infty t^{\delta/2-3/2} e^{-t} e^{-|x|^2/4t} dt \leq C e^{-|x|} \int_1^\infty t^{\delta/2-3/2} dt + C e^{-|x|^2/2} \int_0^1 e^{-1/2t} dt \leq C e^{-|x|}$$

Hence, $R_\delta \in L^p(\mathbb{R})$ with $p < 1/(1 - \delta)$;

Suppose $\delta = 1$, if $x \ll 1$, then

$$R_1(x) = c(\delta) \int_0^\infty t^{-1} e^{-t} e^{-|x|^2/4t} dt \leq C \left(\int_0^{|x|} t^{-1} e^{-|x|/t} dt + \int_{|x|}^\infty t^{-1} e^{-t} dt \right) \leq -C \log |x|$$

and $R_1 \leq C e^{-c|x|}$ when $|x| \rightarrow \infty$. Hence $R_1 \in L^p(\mathbb{R})$ ($p < \infty$);

Suppose $\delta > 1$, then R_δ is bounded and not greater than $C e^{-|x|}$ when $|x| \rightarrow \infty$. Hence $R_\delta \in L^p(\mathbb{R})$ ($p \leq \infty$).

By the same argument we have $R'_{\delta+1} \in L^p(\mathbb{R})$ with $p(1 - \delta) < 1$.

Lemma 3.6. Suppose $u \in C(\mathbb{R}_+; C_{tem}^+)$ is a solution to (18) with $u_0(x) = 0$ for any $x > 0$, then

$$\mathbf{E} \sup_{t \leq T} \left(\int_0^\infty u_t(x) dx \right)^2 \leq C e^{CT}; \quad \mathbf{E} \int_0^T dt \int_0^\infty u_t^2(x) dx \leq C e^{CT}.$$

Proof. Choose

$$\phi_n(x) = \begin{cases} 0 & x \leq 0 \text{ or } x > n \\ \frac{1}{2}[1 + \sin \pi(x - \frac{1}{2})] & 0 < x \leq 1 \\ 1 & 1 < x < n-1 \\ \frac{1}{2}[1 + \cos \pi(x - n + 1)] & n-1 < x \leq n \end{cases}$$

For convenience we omit the subindex n . And all the estimates below are independent with n .

By definition

$$(u_t, \phi) = \int_0^t (u_s, \phi'') ds + \int_0^t (u_s^\gamma \phi, \varphi_k) d\beta_s^k + \int_0^t (u_s h_k, \phi) dw_s^k$$

By Doob's inequality

$$\begin{aligned}
\mathbf{E} \sup_{t \leq T} (u_t, \phi)^2 &\leq C \left\{ \mathbf{E} \int_0^T \sum_k (u_t^\gamma \phi, \varphi_k)^2 dt + \mathbf{E} \int_0^T \sum_k (u_t \phi, h_k)^2 dt + \mathbf{E} \left(\int_0^T |(u_t, \phi'')| dt \right)^2 \right\} \\
&\leq C \left\{ \mathbf{E} \int_0^T dt \int_{\mathbb{R}} u_t^{2\gamma}(x) \phi^2(x) dx + \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} u_t(x) \phi(x) dx \right)^2 + \mathbf{E} \left(\int_0^T (u_s, |\phi''|) dt \right)^2 \right\} \\
&\leq C \left\{ a(T) + \mathbf{E} \int_0^T dt \int_{\mathbb{R}} [u_t(x) \phi(x) + u_t^2(x) \phi^2(x)] dx + \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} u_t(x) \phi(x) dx \right)^2 \right\} \\
&\leq C \left\{ 1 + \mathbf{E} \int_0^T dt \int_{\mathbb{R}} u_t^2(x) \phi^2(x) dx + \int_0^T \mathbf{E} (u_t, \phi)^2 dt \right\}
\end{aligned} \tag{21}$$

Let $v_t = u_t \phi$, v_t satisfies the following equation

$$\partial_t v = \Delta v + \kappa + \phi^{1-\gamma} v^\gamma \varphi_k \dot{\beta}_t^k + v h_k \dot{w}_t^k$$

where $\kappa = -2(u\phi')' + u\phi''$.

$$\begin{aligned}
\|(1 - \Delta)^{-1} \kappa\|_{\mathbb{L}^2(T)}^2 &= \mathbf{E} \int_0^T \int_{\mathbb{R}} |R_2 * \kappa_t(x)|^2 dx dt \\
&\leq C \left\{ \mathbf{E} \int_0^T dt \int_{\mathbb{R}} |R_2' * (u_t \phi')(x)|^2 dx + \mathbf{E} \int_0^T dt \int_{\mathbb{R}} |R_2 * (u_t \phi'')(x)|^2 dx \right\} \\
&\leq C \left\{ \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} |u_t \phi'(x)| dx \right)^2 + \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} |u_t \phi''(x)| dx \right)^2 \right\} \\
&\leq C \left\{ \mathbf{E} \int_0^T dt \int_0^1 u_t(x)^2 dx + \mathbf{E} \int_0^T dt \int_{n-1}^n u_t(x)^2 dx \right\} \\
&\leq a_2(T)
\end{aligned}$$

$$\begin{aligned}
\|(1 - \Delta)^{-1/2} \phi^{1-\gamma} v^\gamma \varphi_k\|_{\mathbb{L}^2(I^2)}^2 &= \mathbf{E} \int_0^T dt \int \sum_k |R_1 * \phi^{1-\gamma} v^\gamma \varphi_k|^2 dx \\
&\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}} R_1^2(x - y) v^{2\gamma}(y) dy \\
&\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^{2\gamma}(x) dx \\
&\leq C K_\epsilon \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t(x) dx + C \epsilon \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx \\
&\leq C K_\epsilon \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 + C \epsilon \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx
\end{aligned}$$

$$\begin{aligned}
\|(1 - \Delta)^{-1/2} v h_k\|_{\mathbb{L}^2(I^2)}^2 &= \mathbf{E} \int_0^T dt \int_{\mathbb{R}} \sum_k |R_1 * v_t h_k(x)|^2 dx \\
&= \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left[\sum_k \left(\int_{\mathbb{R}} R_1(x-y) v_t(y) h_k(y) dy \right)^2 \right] \\
&\leq \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left\{ \int_{\mathbb{R}} \left[|R_1(x-y) v_t(y)|^2 \sum_k h_k^2(y) \right]^{1/2} dy \right\}^2 \\
&\leq C \mathbf{E} \int_0^T dt \|R_1 * v_t\|_2^2 \leq C \mathbf{E} \int_0^T dt \|R_1\|_2^2 \|v_t\|_1^2 \\
&\leq C \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^2
\end{aligned}$$

By [1, Theorem 5.1],

$$\begin{aligned}
\|v\|_{\mathfrak{L}^2(T)}^2 &\leq C \left\{ \|(1 - \Delta)^{-1} \kappa\|_{\mathbb{L}^2(T)}^2 + \|(1 - \Delta)^{-1/2} \phi^{1-\gamma} v^\gamma \varphi_k\|_{\mathbb{L}^2(I^2)}^2 + \|(1 - \Delta)^{-1/2} v h_k\|_{\mathbb{L}^2(I^2)}^2 \right\} \\
&\leq C \left[1 + K'_\epsilon \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 \right] + C\epsilon \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx
\end{aligned}$$

Choose ϵ small, such that $C\epsilon \leq 1/2$. Since $\|v\|_{\mathfrak{L}^2(T)}^2 \geq \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx$, we have

$$\mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx \leq C \left[1 + K'_\epsilon \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 \right] \quad (22)$$

Combining (21), (22) we get

$$\mathbf{E} \sup_{t \leq T} \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 \leq C \left[1 + \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 \right]$$

Using Gronwall's inequality,

$$\mathbf{E} \sup_{t \leq T} \left(\int_{\mathbb{R}} v_t(x) dx \right)^2 \leq C e^{CT}$$

Since our estimates independent with n , we can let $n \rightarrow \infty$, we get

$$\begin{aligned}
\mathbf{E} \sup_{t \leq T} \left(\int_0^\infty u_t(x) dx \right)^2 &\leq C e^{CT} \\
\mathbf{E} \int_0^T dt \int_0^\infty u_t^2(x) dx &\leq C \left[1 + \mathbf{E} \sup_{t \leq T} \left(\int_0^\infty u_t(x) dx \right)^2 \right] \leq C e^{CT}
\end{aligned}$$

□

Lemma 3.7. Suppose $u \in C(\mathbb{R}_+; C_{tem}^+)$ is a solution to (18) satisfying $u_0(x) = 0$ on \mathbb{R}_+ . Then

$$\|u\|_{C^\alpha([0,T] \times \mathbb{R}_+)} < \infty \quad a.s., \quad (23)$$

for some $\alpha \in (0, 1)$ and

$$\mathbf{E} \sup_{t \leq T} \int_0^\infty x u_t(x) dx < \infty \quad (24)$$

Proof. Let

$$\zeta(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2}[1 + \sin \pi(x - \frac{1}{2})] & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

Like the proof of Lemma 3.6 we define $v_t = u_t \zeta$, then v_t satisfies the equation

$$\partial_t v = \Delta v + \kappa + \zeta^{1-\gamma} v^\gamma \varphi_k \dot{\beta}_t^k + v h_k \dot{w}_t^k; \quad v_0 = 0$$

where $\kappa = -2(u\zeta')' + u\zeta''$. Let

$$T_n = n \wedge \inf \left\{ t \geq 0 : \int_0^t \left(\int_{\mathbb{R}} v_s(x) dx \right)^p ds \geq n \right\}.$$

then $T_n \rightarrow \infty$ a.s.. Since R_2 behaviors like $-\log|x|$ near the origin and decreases exponentially, as $|x| \rightarrow \infty$ as before we can prove

$$\|(1 - \Delta)^{-1} \kappa\|_{\mathfrak{L}^p(T)}^p < \infty \quad (\forall p \in \mathbb{N}).$$

We claim for any $p \geq 2$,

$$\mathbf{E} \int_0^T \|v_t\|_p^p dt < \infty. \quad (25)$$

Let $p_k = 2/\gamma^k$, if we have

$$\mathbf{E} \int_0^T \|v_t\|_{p_k}^{p_k} dt < \infty.$$

then

$$\begin{aligned} \|(1 - \Delta)^{-1/2} \zeta^{1-\gamma} v^\gamma \varphi_k\|_{\mathbb{L}^{p_{k+1}}(l^2)}^{p_{k+1}} &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} R_1^2(x - y) v_t^{2\gamma}(y) dy \right]^{p_{k+1}/2} \\ &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^{p_k}(x) dx < \infty \\ \|(1 - \Delta)^{-1/2} v h_k\|_{\mathbb{L}^{p_{k+1}}(l^2)}^{p_{k+1}} &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left\{ \sum_k \left[\int_{\mathbb{R}} R_1(x - y) v_t h_k(y) dy \right]^2 \right\}^{p_{k+1}/2} \\ &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} R_1(x - y) v_t(y) dy \right]^{p_{k+1}} \\ &\leq C \mathbf{E} \int_0^T dt \left(\int_{\mathbb{R}} v_t(x) dx \right)^{p_{k+1}} < \infty \end{aligned}$$

Hence, if (25) holds, by [1, Theorem 5.1], we obtain

$$\mathbf{E} \int_0^T \|v_t\|_{p_{k+1}}^{p_{k+1}} dt \leq \|v\|_{\mathbb{L}^{p_{k+1}}(T)}^{p_{k+1}} < \infty$$

Hence, for any $p \geq 2$, inequality (25) holds. Like the argument in the Lemma 1.5 of [2], choose $1/2 < \delta < 1$, for any $p \geq 2$,

$$\begin{aligned} \left(\int_{\mathbb{R}} |R_{\delta+1} * \kappa_t(x)|^p dx \right)^{1/p} &\leq C \left(\int_{\mathbb{R}} |R'_{\delta+1} * (\phi' u_t)(x)|^p dx \right)^{1/p} + C \left(\int_{\mathbb{R}} R_{\delta+1} * (\phi'' u_t)(x) dx \right)^{1/p} \\ &\leq C(\|R'_{\delta+1}\|_p + \|R_{\delta+1}\|_p) \int_{[0,1] \cup [n-1,n]} u_t(x) dx \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E} \int_0^T dt \int_{\mathbb{R}} |R_{\delta+1} * \kappa_t(x)|^p dx &\leq C \mathbf{E} \int_0^T dt \left(\int_{[0,1] \cup [n-1,n]} u_t(x) dx \right)^p < \infty \\ \|(1 - \Delta)^{-\delta/2} \phi^{1-\gamma} v^\gamma \varphi_k\|_{\mathbb{L}^p(l^2)}^p &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} R_\delta^2(x-y) v_t^{2\gamma}(y) dy \right]^{p/2} \\ &\leq C \|R_\delta\|_2^{p/2} \mathbf{E} \int_0^T dt \int_{\mathbb{R}} v_t^{\gamma p}(x) dx < \infty \\ \|(1 - \Delta)^{-\delta/2} v h_k\|_{\mathbb{L}^p(l^2)}^p &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left\{ \sum_k \left[\int_{\mathbb{R}} R_\delta(x-y) v_t h_k(y) dy \right]^2 \right\}^{p/2} \\ &\leq C \mathbf{E} \int_0^T dt \int_{\mathbb{R}} dx \left[\int_{\mathbb{R}} R_\delta(x-y) v_t(y) dy \right]^p \\ &\leq C \|R_\delta\|_1^p \mathbf{E} \int_0^T dt \|v_t\|_p^p < \infty \end{aligned}$$

By [1, Theorem 5.1], we have $v \in \mathcal{H}_p^{1-\delta}$. By choosing for instance $\delta = 0.6$, $p = 33$, we have $v \in \mathcal{H}_p^{1-\delta}(T)$, and by Sobolev's embedding theorem, $C^{1/10}([0, T] \times \mathbb{R}) \subset \mathcal{H}_p^{1-\delta}(T)$ we get

$$\|u\zeta\|_{C^{1/10}([0,T] \times \mathbb{R})} < \infty \quad a.s.$$

Now we prove (24). Choose $\eta_n(x) \in C_c(\mathbb{R})$ and $\text{supp } \eta_n \in \mathbb{R}_+$, $\eta_n(x) = x$ when $x \in [1, n]$ and $\sup_{x,n} |\eta_n''(x)| < \infty$.

$$\begin{aligned} 0 &\leq \int_1^n x u_t(x) dx \leq \int_{\mathbb{R}} \eta_n(x) u_t(x) dx \\ &= \int_0^t \int_{\mathbb{R}} u_s(x) \eta_n''(x) dx ds + M_t^n \\ [M^n]_t &= \int_0^t u_s^{2\gamma}(x) \eta_n^2(x) dx ds + \sum_k \int_0^t ds \left(\int_{\mathbb{R}} u_s(x) h_k(x) \eta_n(x) dx \right)^2 \\ &\leq C \int_0^t \left(\int_{\mathbb{R}} u_s(x) dx \right)^2 ds + \int_0^t \int_{\mathbb{R}} u_s^2(x) dx ds \in L^1(\mathbf{P}) \end{aligned}$$

Hence M_t^n is a martingale, taking expectation and let $n \rightarrow \infty$, we get

$$\mathbf{E} \sup_{t \leq T} \int_0^\infty x u(t, x) dx \leq C \mathbf{E} \sup_{t \leq T} \int_0^\infty u_t(x) dx < \infty$$

□

The proof of Theorem 3.5 follows the idea of [2], we present here for reader's convenience.

Proof Of Main Theorem. We follow the proof in [2].

Step 1. For $\psi \in C_c(\mathbb{R})$, if $\psi'' = \nu$ is a finite measure on \mathbb{R} , then equation (18) also holds. (see [2, Lemma 3.1])

Step 2. On the set $\{\omega : \int_0^T u_s(0, \omega) ds = 0\}$, $u(t, x) = 0 \ \forall x > 0, t \in [0, T]$.

Let $\psi_n = x_+ \phi(x/n)$,

$$0 \leq \int_0^\infty \psi_n(x) u_t(x) dx = \int_0^t u_s(0) ds + \int_0^t \int_0^\infty u_s(x) \psi_n''(x) dx ds + M_t^n \quad (26)$$

where M_t^n is a local martingale with

$$[M^n]_t = \int_0^t \int_0^\infty \psi_n^2(x) u_s^{2\gamma}(x) dx ds + \sum_k \int_0^t ds \left(\int_{\mathbb{R}} \psi_n(x) h_k(x) u_s(x) dx \right)^2 \in L^1(\mathbf{P}).$$

Hence M_t^n is a martingale, so $\mathbf{E}|M_\tau^n| = 2\mathbf{E}(M_\tau^n)^-$ for any bounded stopping time τ . Let

$$V_t^n = \int_0^t u_s(0) ds + \int_0^t \int_0^\infty u_s(x) \psi_n''(x) dx ds.$$

Using (26),

$$(M_t^n)^- \leq V_t^n \leq C \left\{ \int_0^t u_s(0) ds + \frac{1}{n} \int_0^t \int_0^\infty u_s(x) dx ds + \frac{1}{n^2} \int_0^t \int_0^\infty x u_s(x) dx ds \right\}$$

Hence for any bounded stopping time τ ,

$$\mathbf{E}|M_\tau^n| = 2\mathbf{E}(M_\tau^n)^- \leq 2\mathbf{E}V_\tau^n$$

By the generalized Ito's inequality, we get for any $0 < \alpha < 1$ and any bounded stopping time τ ,

$$\begin{aligned} \mathbf{E} \left(\int_0^\tau \int_0^\infty \psi_n^2(x) u_s^{2\gamma}(x) dx ds \right)^{\alpha/2} &\leq \mathbf{E}[M_\tau^n]^{\alpha/2} \leq C \mathbf{E}(\sup_{s \leq \tau} |M_s^n|)^\alpha \\ &\leq C \mathbf{E}(V_\tau^n)^\alpha \leq C \mathbf{E}V_\tau^n \\ &\leq C \left\{ \mathbf{E} \int_0^\tau u_s(0) ds + \frac{1}{n} \mathbf{E} \int_0^\tau \int_0^\infty u_s(x) dx ds \right. \\ &\quad \left. + \frac{1}{n^2} \mathbf{E} \int_0^\tau \int_0^\infty x u_s(x) dx ds \right\} \end{aligned}$$

Let $n \rightarrow \infty$, we get

$$\mathbf{E} \left(\int_0^\tau \int_0^\infty x^2(x) u_s^{2\gamma}(x) dx ds \right)^{\alpha/2} \leq C \mathbf{E} \int_0^\tau u_s(0) ds$$

Now let $\tau = \inf\{t : \int_0^t u_s(0) ds > 0\}$, we obtain the conclusion.

Step 3. Following the proof of Lemma 2.1 of [2] one can show: If $\gamma \in [\frac{1}{2}, 1)$, then for any $p, q > 0$, $0 < r \leq 1$, $0 < \alpha < 1$ there exists a point $x \in [r, 2r]$ such that

$$\mathbf{P} \left(\int_0^T u_s(x) ds \geq p \right) \leq \mathbf{P} \left(\int_0^T u_s(0) ds \geq q \right) + C r^{-3\alpha/2} \left(\frac{q}{p^\gamma} \right)^\alpha$$

and here C is independent with p, q, r .

Step 4. Now the Theorem can be prove just as Theorem 1.7 of [2].

□

REFERENCES

- [1] Nicolai V Krylov. An analytic approach to spdes. *Stochastic partial differential equations: six perspectives*, 64:185–242, 1999.
- [2] NV Krylov. On a result of c. mueller and e. perkins. *Probability theory and related fields*, 108(4):543–557, 1997.
- [3] Carl Mueller and Edwin A Perkins. The compact support property for solutions to the heat equation with noise. *Probability Theory and Related Fields*, 93(3):325–358, 1992.
- [4] Leonid Mytnik et al. Superprocesses in random environments. *The Annals of Probability*, 24(4):1953–1978, 1996.
- [5] Leonid Mytnik and Jie Xiong. Local extinction for superprocesses in random environments. *Electronic Journal of Probability*, 12:1349–1378, 2007.
- [6] Edwin Perkins. Part ii: Dawson-watanabe superprocesses and measure-valued diffusions. *Lectures on probability theory and statistics*, pages 125–329, 2002.
- [7] Tokuzo Shiga. Two contrasting properties for solutions of one dimensional stochastic partial differential equations. *Canadian Mathematical Journals*, 46:415–437, 1994.

Guohuan Zhao

School of Mathematical Sciences, Peking University,

Beijing, 100871, P.R. China.

Email: zhaogh@pku.edu.cn